Outline of General Relativity
and an Investigation of
Super-Light Expansion Phenomina

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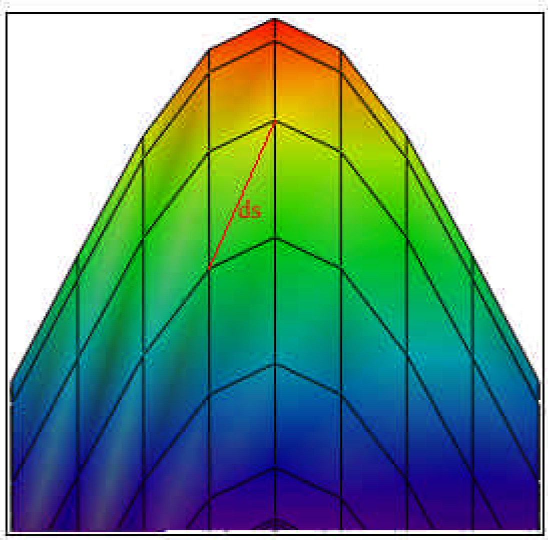
Brief History of Relativity

- Michelson-Morley Experiment (1887)

- Special Relativity (1905)
  1. No absolute reference frame exists
  2. Speed of light with respect to inertial frames is constant

- General Relativity (1916)
  1. Principle of covariance (form invariance)
  2. Principle of equivalence (gravitation = inertia)
  3. $G_{\mu\nu} = 0$ (Law of Gravitation)

The Metric Tensor, g_{ij} for a surface in 3-space



Equation of surface:

$f(x_1, x_2, x_3)$

Note: Surface coordinates (U_1, U_2) have been constructed and

$x_i = x_i(U_1, U_2)$

$$ds^2 := dx_i \cdot dx_i$$

Or

$$ds^2 := g_{ij} \cdot du_i \cdot du_j$$

$$g_{ij} := \sum_{\alpha = 1}^{3} \frac{\partial}{\partial U_i} x_\alpha \cdot \frac{\partial}{\partial U_j} x_\alpha$$

What is $G_{\mu\nu} = 0$ and how (generally) do we get it?

1. Replace "gravitational field" concept with concept of "distortion of geometry".

But how?

2. Find most general form of geometry

- Finsler Geometry? (No. ruled out by requirement for rigid body rotation form invariance.)
- Riemannian Geometry? (Apparently)

$$ds^2 := \sum_{(i,j)} g_{ij}\, dx_i\, dx_j$$

Where g_{ij} are functions of $x_1, x_2, x_3, ..., x_n$

THEN?

3. Restrict the Riemannian geometry

- Make ds^2 invariant under coordinate changes
- make empty space "flat". ($g_{\mu\nu}$ constant)
- Satisfy Special Relativity conditions (Lorentz Transformation)
- Covariance of laws of physics with respect to coordinate transformations. (No favored coordinate system.)

Now What?

4. Apply the above restrictions

Eventually leads to the condition:

$R^{\varepsilon}_{\ \mu\nu\sigma} = 0$ (Rieman-Christoffel curvature tensor)

So What is $R^{\varepsilon}_{\ \mu\nu\sigma} = 0$ and how does it relate to $G_{\mu\nu} = 0$?

5. It is:

$$R^{\varepsilon}_{\ \mu\nu\sigma} = \{\mu\nu,\alpha\}\{\alpha\nu,\varepsilon\} - \{\upsilon\varpi,\alpha\}\{\alpha\sigma,\varepsilon\} \frac{\partial}{\partial x_\nu} \{\mu\nu,\varepsilon\} - \frac{\partial}{\partial x_\sigma} \{\mu\nu,\varepsilon\}$$

$$\text{where} \qquad \{\mu\sigma,\alpha\} = \frac{1}{2} \cdot g^{\alpha\lambda} \left(\frac{\partial}{\partial x_\sigma} g_{\mu\lambda} + \frac{\partial}{\partial x_\mu} g_{\sigma\lambda} - \frac{\partial}{\partial x_\lambda} g_{\mu\sigma} \right)$$

Now it turns out that $R^{\varepsilon}{}_{\mu\nu\sigma} = 0$ is <u>too</u> restrictive. It makes the g_{ij} constant <u>always</u> and does not allow for a "gravitational field." So Einstein set $\sigma = \varepsilon$ (i.e. performed a "tensor contraction" operation) to obtain

$$G_{\mu\nu} = R^{\varepsilon}{}_{\mu\nu\varepsilon} = 0$$

as the "law of gravitation."

Now that we know $G_{\mu\nu} = 0$, what good is it?

$G_{\mu\nu}$ is a symmetric 4x4 matrix which yields 10 simultaneous partial differential equations which can be solved for the g_{ij} - (maybe!)

When you have the g_{ij} values you can find the line element, ds, which lets you solve practical problems.

The Schwarzschild solution **(The gravitational field of an isolated particle)**

1. Start from:

$$ds^2 := -dr^2 - r^2\,d\theta^2 - r^2\sin(\theta)^2\,d\phi^2 + c^2\,dt^2$$

2. Generalize slightly:

$$ds^2 := -A(r)\,dr^2 - B(r)\cdot r^2\cdot d\theta^2 - C(r)\sin(\theta)^2\,d\phi^2 + D(r)\,dt^2$$

3. Symmetry conditions and the judicious choice of radial coordinates leads to:

$$ds^2 := -e^{\lambda(r)}\,dr^2 - r^2\,d\theta^2 - r^2\sin(\theta)^2\,d\phi^2 + e^{\nu(r)}\,dt^2$$

4. Now note:

$$ds^2 := \begin{pmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & e^{\nu} \end{pmatrix} \cdot \begin{pmatrix} dr^2 \\ d\theta^2 \\ d\phi^2 \\ dt^2 \end{pmatrix}$$

Note the similarity with:

$$ds^2 := \sum_{(i,j)} \left(g_{ij} dx_i \cdot dx_j \right)$$

So we see that the g_{ij} are the elements of the matrix given above.

Now remember $G_{\mu\nu} = 0$? Well it means $G_{11} = 0$, $G_{12} = 0$, etc.

Now calculation shows:

$$G_{\mu\nu} := -\left(\frac{\partial}{\partial x_\alpha} \right) \mu\nu,\alpha\} + \{\mu\alpha,\beta\}\ \{\nu\beta,\alpha\} + \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \log\sqrt{-g}$$

$$- \{\mu\nu,\alpha\} \quad \frac{\partial}{\partial x_\alpha} \log\sqrt{-g}$$

So, for G_{11} we have:

$$G_{11} := -\frac{\partial}{\partial x_\alpha} \quad \{11,\alpha\} + \{1\alpha,\beta\}\ \{1\beta,\alpha\} + \frac{\partial^2}{\partial r^2} \log\sqrt{-g}$$

$$- \{11,\alpha\} \ \frac{\partial}{\partial x_\alpha} \log\sqrt{-g}$$

$$\text{Now }\{11,1\} = \frac{1}{2}\cdot g^{1\lambda}\left(\frac{\partial}{\partial x_1}g_{1\lambda} + \frac{\partial}{\partial x_1}g_{1\lambda} - \frac{\partial}{\partial x_\lambda}g_{11}\right)$$

$$= \frac{1}{2}g^{11}\cdot\left(\frac{\partial}{\partial x_1}g_{11} + \frac{\partial}{\partial x_1}g_{11} - \frac{\partial}{\partial x_\lambda}g_{11} + 0 + 0 + 0\right)$$

$$= \frac{-1}{2e^{\lambda}}\cdot\frac{\partial}{\partial r}\left(-e^{\lambda}\right)$$

$$= \frac{1}{2}\cdot\frac{e^{\lambda}}{e^{\lambda}}\cdot\frac{\partial}{\partial r}\lambda$$

$$= \frac{1}{2}\cdot\frac{d}{dr}\lambda$$

Now, many of the $G_{\mu\nu}$ derived as above, are identically zero but the surviving system of differential equations is soluable and we find eventually that

$$\lambda := -\nu$$

and

$$e^{\lambda} := \gamma = 1 - 2\frac{m}{r} \qquad \left(\text{note that} \quad m := \frac{GM}{c^2}\right)$$

So we have the Swarzchild solution which yields the line element:

$$ds^2 := \frac{1}{\gamma}\cdot dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + \gamma dt^2$$

We can now exploit the above expression to obtain several

Predictions of General Relativity

Prediction I **The path of a planet is a Roulette (a precessing ellipse) rather than an ellipse.**

By using the calculus of variations, it is possible to obtain the equations of a geodesic in Riemann space. The geodesic equation is:

$$\frac{d^2}{ds^2}x_\alpha \quad + \ \{\mu\nu,\alpha\} \quad \frac{d}{ds}x_\mu \cdot \frac{d}{ds}x_\nu$$

Applying this to the Schwarzschild line element gives the equation for the path taken by an object in space. The path is given by:

$$\frac{d^2}{d\phi^2}u + u := \frac{m}{h^2} + 3mu^2$$

where $\quad h := r^2\frac{d}{ds}\phi \quad$ and $\quad u := \frac{1}{r}$

Compare with the Newtonian result:

$$\frac{d^2}{d\phi^2}u + u := \frac{Gm}{H^2}$$

where $\quad H := r^2\frac{d}{dt}\phi \quad$ and $\quad u := \frac{1}{r}$

Converting the relativistic h to Newtonian terms for further comparison:

$m := \frac{GM}{c^2} \quad$ and $\quad \frac{d}{ds}\phi := \frac{d}{dt}\phi \cdot \frac{d}{ds}t$

which is approximately equal to $\frac{1}{c} \cdot \frac{d}{dt}\phi$

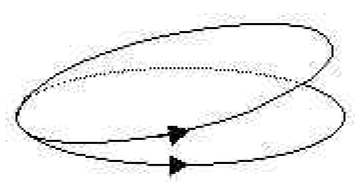
so h^2 is approximately:

$$\frac{GM}{r^4\left(\frac{d}{dt}\phi\right)^2} := H^2$$

I.e., h = H approximately.

So we see that he Relativistic path is essentially the same as the Newtonian path except for the correction term $2mu^2$.

The differential equation for the path can be solved approximately by the method of successive approximation.



The path equations predict that the orbit of Mercury will shift 42.9 seconds of arc per century.

This degree of orbit precession was known before Relativity provided the explanation. (Astronomers had been looking for "Vulcan" because of this anomaly.)

Prediction II **Light passing near the Sun will be deflected through an angle of 1.75 arc-seconds.**

Returning to the Relativistic path differential equation:

$$\frac{d^2}{d\phi^2}u + u := \frac{m}{h^2} + 3mu^2$$

where $h := r^2\frac{d}{ds}\phi$ and $u := \frac{1}{r}$

We observe, from Special Relativity:

$$ds^2 := c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Or

$$\left(\frac{d}{dt}s\right)^2 := c^2 - \left(\frac{d}{dt}x\right)^2 - \left(\frac{d}{dt}y\right)^2 - \left(\frac{d}{dt}z\right)^2$$

$$= \quad c^2 - v^2$$

For a light ray, ds = 0. Now if ds = 0, then

$$v^2 := c^2$$

I.e.,

$$v := c$$

From above, this implies that

$$h = \lim_{ds\to 0} r^2\frac{d}{ds}\phi \; := \infty$$

and thus $\frac{m}{h^2}$ vanishes

yielding:

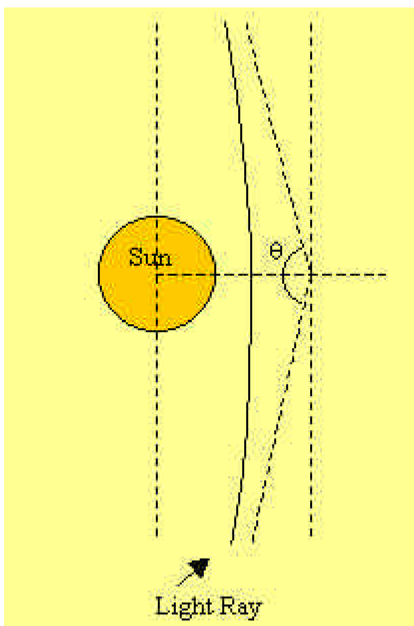
$$\frac{d^2}{d\phi^2}u + u := 3mu^2$$ as the path of a light ray.

solving

$$\frac{d^2}{d\phi^2}u + u := 3mu^2$$

by successive approximation yields the path equation.
In rectangular coordinates, it is:

$$x := R - \frac{m}{R} \cdot \frac{x^2 + 2 \cdot y^2}{\sqrt{x^2 + y^2}}$$



Asymptotes are found by setting y >> x. The two solutions are:

$$x := R + \frac{m}{R}2y$$

and

$$x := R - \frac{m}{R}2y$$

and the angle between the asymptotes is:

$$\theta := \tan^{-1}4\frac{m}{R}$$

or θ is approximately

$$4\frac{m}{R}$$

The deflection was confirmed during the 29 May 1919 Solar Eclipse.
Star images in the vicinity of the sun appeared to be outwardly displaced on the photographic plates by the predicted 1.75 arc-seconds.

Prediction III

The gravitational red-shift. The characteristic wavelength of light is shifted toward the red end of the spectrum when the light originates in an intense gravitational field.

Returning to the Schwarzschild line element:

$$ds^2 := \frac{1}{\gamma} \cdot dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + \gamma dt^2$$

and setting dr = dθ = dϕ = 0, we have:

$$ds^2 := \gamma dt^2$$

This says that time is distorted in a gravitational field.

Consider an atom on the sun radiating energy and one on the Earth radiating energy. By the principle of covariance, when viewed by a <u>close</u> observer, measurement of the frequency should be identical. But an observer on Earth will see a different frequency produced by an atom on the Sun.

$$ds_1^{\,2} := dt_1^{\,2}$$

$$ds_2^{\,2} := \gamma dt_2^{\,2}$$

ds is invariant so:

$$ds_1 := ds_2$$

and so dt_1, the period, as seen on the earth is:

$$dt_1^{\,2} := \frac{1}{\gamma} dt_2^{\,2}$$

Now $\frac{1}{\gamma} := \frac{1}{1 - 2\frac{m}{r}}$ ∎

which at the surface of the sun is 1.00000212. Difficult to confirm for the Sun but was confirmed for the white dwarf companion of Sirius where the effect is 30 times stronger.

This effect was unexpected prior to General Relativity.

The result has been observed on earth (Pound & Rebka 1960) using the mossbauer effect.

Prediction IV Radar Echo Delay. (Start from path equation for light ray, integrate along the path to obtain the "true" time for the echo to be heard. Delay of 240 microseconds for Mercury near superior conjunction. Difficult to test. Confirmed by transponder.

Prediction V Precession of a gyroscope in Earth orbit. Complicated and difficult to quantify. Best calculations show the spin vector should change after many revolutions about the Earth.